math 251 - week 10 - Ch6 Inner Product Space.

الجداء الداخلي

def. $\langle u, v\rangle$ there are 4 Axioms :- $\Rightarrow$ rules

① $\langle u, v\rangle = \langle v, u\rangle$

② $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

③ $c\langle u, v\rangle = \langle cu, v\rangle$

④ $\langle u, u\rangle \geqslant 0$ and $\langle u, u\rangle = 0$ iff $u = 0$.

Ex) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be a vectors in R^2, verify that the Eucledian inner Product $\langle u, v\rangle = 3u_1 v_1 + 2u_2 v_2$ satisify the four Inner Product Axioms.

Sol. ① $\langle u, v\rangle = 3u_1 v_1 + 2u_2 v_2$

$= 3v_1 u_1 + 2v_2 u_2$

$= \langle v, u\rangle$ satisify

② Let (assume) $w = (w_1, w_2)$, then

$$\langle \underline{\underline{u}}+v, \underline{\underline{w}}\rangle = 3(u_1+v_1).w_1 + 2(u_2+v_2).w_2$$

$$= 3u_1w_1 + 3v_1w_1 + 2u_2w_2 + 2v_2w_2$$

$$= (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2)$$

$$= \langle u, w\rangle + \langle v, w\rangle$$

③ Assume any constant,

$$\langle \underbrace{cu}_{u}, \underbrace{v}_{v} \rangle = 3(cu_1)v_1 + 2(c.u_2)v_2$$
$$= c(3u_1v_1) + c(2u_2v_2)$$
$$= c[3u_1v_1 + 2u_2v_2]$$
$$= c\langle u, v\rangle$$

④

$$\langle u, u \rangle = 3(u_1, u_1) + 2(u_2 \cdot u_2)$$
$$= 3u_1^2 + 2u_2^2 \geq 0$$

$\therefore \langle u, u\rangle$ is an inner Product space.

<u>Ex</u> $\langle u, v\rangle = u_1v_1 + 2u_2v_2$

<u>Sol.</u> ① $\langle u, v\rangle = u_1v_1 + 2u_2v_2$
$$= v_1u_1 + 2v_2u_2$$
$$= \langle v, u\rangle$$

② let $w = (w_1, w_2)$ then

$$\langle \underbrace{u}_{u}, \underbrace{v+w}_{v} \rangle = u_1(v_1 + w_1) + 2u_2(v_2 + w_2)$$
$$= \underline{u_1v_1} + u_1w_1 + \underline{2u_2v_2} + 2u_2w_2$$
$$= (u_1v_1 + 2u_2v_2) + (u_1w_1 + 2u_2w_2)$$

$= \langle u, v \rangle + \langle u, w \rangle$

③
$$c\langle u, v\rangle = c\langle u_1 v_1 + 2u_2 v_2\rangle$$
$$= (cu_1)v_1 + 2(cu_2)v_2$$
$$= \langle cu, v\rangle$$

④ $\langle u, u\rangle = u_1^2 + 2u_2^2 \geq 0$

$\langle u, u\rangle = 0 \Rightarrow u_1^2 + 2u_2^2 = 0 \Rightarrow u_1 = u_2 = 0$

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* Distance between u and v:

$$d(u, v) = \|u - v\| = \sqrt{\langle u - v, u - v\rangle}$$

* Angle between the two non-Zero vectors:

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} \qquad 0 \leq \theta \leq \pi$$

Note: if $u \perp v$, then $\langle u, v\rangle = 0$ orthogonal

if $\|u\| = 1$, then v is called a unit vectors

Orthonormal
↓ orthogonal
→ the norm of vector $= 1$ (must be unit vector).

Ex/ Show that the following set is orthonormal basis.

$$S = \left\{ \underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}_{u_1}, \underbrace{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)}_{u_2}, \underbrace{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)}_{u_3} \right\}.$$

Sol. show that orthogonal :-

$$u_1 \cdot u_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot \left(\frac{-\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{6}\right)$$

dot product

$$= \frac{1}{\sqrt{2}} \cdot \frac{-\sqrt{2}}{6} + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{6} + 0 \cdot \frac{2\sqrt{2}}{6}$$

$$= \cancel{\frac{-1}{6}} + \cancel{\frac{1}{6}} + 0$$

$$u_1 \cdot u_2 = 0 \Rightarrow \text{orthogonal} \quad u_1 \perp u_2$$

$$u_1 \cdot u_3 = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} + \frac{1}{\sqrt{2}} \cdot \frac{-2}{3} + 0 \cdot \frac{1}{3}$$

$$= \frac{\sqrt{2}}{3} + \frac{-\sqrt{2}}{3} + 0$$

$$u_1 \cdot u_3 = 0 \Rightarrow \text{orthogonal} \quad u_1 \perp u_3$$

$$u_2 \cdot u_3 = \frac{-\sqrt{2}}{6} \cdot \frac{2}{3} + \frac{\sqrt{2}}{6} \cdot \frac{-2}{3} + \frac{2\sqrt{2}}{3} \cdot \frac{1}{3}$$

$$= \frac{-\sqrt{2}}{9} + \frac{-\sqrt{2}}{9} + \frac{2\sqrt{2}}{9}$$

$u_2 \cdot u_3 = 0 \Rightarrow$ orthogonal $u_2 \perp u_3$

For normal:

$$\|u_1\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = \underline{\underline{1}}.$$

$$\|u_2\| = \sqrt{\left(\frac{-\sqrt{2}}{6}\right)^2 + \left(\frac{\sqrt{2}}{6}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2} = \underline{\underline{1}}.$$

$$\|u_3\| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \underline{\underline{1}}.$$

Now we can say that S is a orthonormal.

the set S

Note that: the vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}$ are Unit Vectors.

so, that means S is Orthonormal.

bcz the norm equals $= \underline{\underline{1}}$

Ex] Let $\mathbb{R}^3$ have a Eucledian inner product space, Find the cosine angle between $u=(-1,6,2)$ and $v=(4,3,-5)$??

Sol. We now that $\cos\theta = \dfrac{\langle u, v\rangle}{\|u\|\,\|v\|}$

$\langle u, v\rangle = (-1)(4) + (6)(3) + (2)(-5)$

$\langle u, v\rangle = -4 + 18 - 10$

$\langle u, v\rangle = 4$

$\|u\| = \sqrt{(-1)^2 + 6^2 + 2^2} = \sqrt{1+36+4} = \sqrt{41}$

$\|v\| = \sqrt{4^2 + 3^2 + (-5)^2} = \sqrt{50}$

Now:

$$\cos\theta = \frac{4}{\sqrt{41}\,\sqrt{50}} = \frac{4}{\sqrt{2050}}$$

Ex) Let $\mathbb{R}^3$ have a Eucledian inner Product for What value of $\underline{\underline{k}}$ if u and v are orthogonal.

ⓐ $u = (2,1,3)$ $v = (1,7,k)$.

ⓑ $u = (k,k,1)$ $v = (k,5,6)$.

Sol. ⓐ Since u and v are orthogonal, then

$$\langle u, v \rangle = 0$$

$$\langle (2,1,3) \cdot \langle 1,7,k \rangle \rangle = 0$$

$$2.1 + 1.7 + 3.k = 0$$

$$2 + 7 + 3k = 0$$

$$9 + 3k = 0$$

$$\frac{\cancel{3}k}{\cancel{3}} = \frac{-9}{3}$$

$$\boxed{k = -3}$$

(b) Since u & v are orthogonal, then

$\langle u, v \rangle = 0$

$\langle (k,k,1) \cdot (k,5,6) \rangle = 0$

$k^2 + 5k + 6 = 0$

$(k+2)(k+3) = 0$

$\boxed{k = -2, -3}$

Ex) $p = 1 - 2x + 3x^2$, $q = 3 + x^2$

① find $\|p\|$ and $\|q\|$?? ② find $\langle p, q \rangle$??

Sol.

① $\|p\| = \sqrt{1^2 + (-2)^2 + (3)^2}$

$\|q\| = \sqrt{(3)^2 + (1)^2}$

② $\langle p, q \rangle = \langle 1 - 2x + 3x^2 . (3 + x^2) \rangle$

$= 1.3 + (-2 . 0) + 3.1$

$= 3 + 3$

$\langle p, q \rangle = 6$